Merten's Theorem

Though we have yet to prove any asymptotic results on $\pi(x)$ we can prove such results for *weighted* sums over primes.

Theorem 2.22 ***Merten's Theorem*** *As $x \to \infty$ we have*

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + O(1)\,, \tag{16}$$

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1)\,, \tag{17}$$

$$\sum_{p \leq x} \frac{1}{p} = \log\log x + O(1)\,. \tag{18}$$

The last result is an improvement of Theorem 1 in Chapter 1, a lower bound on the summation..

Proof. The integral part of $y \in \mathbb{R}$ and satisfies $y - 1 < [y] \leq y$. Thus, since $\Lambda(n) \geq 0$ for all n, we have

$$\sum_{n \leq x} \Lambda(n) \left(\frac{x}{n} - 1\right) \leq \sum_{n \leq x} \Lambda(n) \left[\frac{x}{n}\right] \leq \sum_{n \leq x} \Lambda(n)\, \frac{x}{n}.$$

This can be rearranged as

$$0 \leq\ x \sum_{n \leq x} \frac{\Lambda(n)}{n} - \sum_{n \leq x} \Lambda(n) \left[\frac{x}{n}\right]\ \leq\ \sum_{n \leq x} \Lambda(n) = \psi(x) = O(x)\,,$$

by a weak form of Chebyshev's Theorem. Thus

$$\begin{aligned} x \sum_{n \leq x} \frac{\Lambda(n)}{n} &= \sum_{n \leq x} \Lambda(n) \left[\frac{x}{n}\right] + O(x) \\ &= (x \log x - x + O(\log x)) + O(x)\,, \end{aligned}$$

by Lemma 2.12. Dividing by x gives the **first result**

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + O(1)\,. \tag{19}$$

For the **second result,** (17), we just need check that the contribution to the sum in the left hand side of (19) from the powers of primes can be absorbed into the $O(1)$ error term. Split off the prime powers as in

$$\sum_{n\le x}\frac{\Lambda(n)}{n} = \sum_{p^k\le x}\frac{\log p}{p^k} = \sum_{p\le x}\frac{\log p}{p} + \sum_{\substack{p^k\le x\\ k\ge 2}}\frac{\log p}{p^k}.$$

In the second sum here, for each prime p the range of k is $2 \le k \le (\log x)\,/\,(\log p)$. At the cost of making it larger we extend this to **all** $k \ge 2$, and sum the resulting geometric series as

$$\sum_{\substack{p^k\le x\\ k\ge 2}}\frac{\log p}{p^k} \le \sum_{p\le x}\log p\sum_{k\ge 2}\frac{1}{p^k} \le \sum_{p\le x}\log p\frac{1/p^2}{1-1/p} = \sum_{2\le p\le x}\frac{\log p}{p^2-p},$$

on summing the geometric series. At the cost of making it larger again we extend this sum over **primes** $2 \le p \le x$ to a sum over all **integers** $n \ge 2$. That is,

$$\sum_{2\le p\le x}\frac{\log p}{p^2-p} \le \sum_{2\le n}\frac{\log n}{n\,(n-1)} = O(1)$$

since this is a convergent series (by comparison, for example, with $\sum_{n\ge 1} 1/n^{3/2}$ which can be summed using Corollary 2.10). Hence we get the second result

$$\sum_{p\le x}\frac{\log p}{p} = \sum_{n\le x}\frac{\Lambda(n)}{n} + O(1) = \log x + O(1)$$

by using the first result.

Finally, the method of Partial Summation gives

$$\begin{aligned}
\sum_{2\le p\le x}\frac{1}{p} &= \sum_{2\le p\le x}\frac{\log p}{p}\frac{1}{\log p} \\
&= \sum_{2\le p\le x}\frac{\log p}{p}\left(\frac{1}{\log x}-\left(\frac{1}{\log x}-\frac{1}{\log p}\right)\right) \\
&= \frac{1}{\log x}\sum_{2\le p\le x}\frac{\log p}{p}+\sum_{2\le p\le x}\frac{\log p}{p}\int_p^x\frac{dt}{t\log^2 t} \\
&= \frac{1}{\log x}\sum_{2\le p\le x}\frac{\log p}{p}+\int_2^x\sum_{2\le p\le t}\frac{\log p}{p}\frac{dt}{t\log^2 t} \qquad (20) \\
&= \frac{\log x+O(1)}{\log x}+\int_2^x\frac{(\log t+O(1))}{t\log^2 t}dt \\
&\qquad \text{by part ii} \qquad (21) \\
&= O(1)+\log\log x+O\left(\int_2^x\frac{1}{t\log^2 t}dt\right).
\end{aligned}$$

This final integral converges, (change variables to $w=\log t$ to see this more obviously). Thus we have the final result. ■

Merten's Theorems extended

In Theorem 22 we proved that

$$\sum_{p\leq x}\frac{1}{p}=\log\log x+O(1)\,, \tag{22}$$

for real x. We earlier proved

$$\sum_{n\leq x}\frac{1}{n}=\log x+O(1)\,,$$

which we then extended to

$$\sum_{n\leq x}\frac{1}{n}=\log x+\gamma+O\left(\frac{1}{x}\right)\,,$$

with the Euler-Mascheroni constant γ.

It is straightforward to extend (22) in the same way.

Theorem 2.23 *There exists a constant D such that*

$$\sum_{p\leq x}\frac{1}{p}=\log\log x+D+O\left(\frac{1}{\log x}\right)$$

as $x\to\infty$.

Proof Write Merten's result as

$$\sum_{p\leq x}\frac{\log p}{p}=\log x+\varepsilon(x)$$

where $|\varepsilon(x)|<C$ for some constant C. Partial summation gives

$$\begin{aligned}\sum_{2\leq p\leq x}\frac{1}{p} &= \frac{\log x+\varepsilon(x)}{\log x}+\int_2^x\frac{(\log t+\varepsilon(t))}{t\log^2 t}dt\\ &= 1+O\left(\frac{1}{\log x}\right)+\log\log x-\log\log 2+\int_2^x\frac{\varepsilon(t)}{t\log^2 t}dt\\ &= \log\log x+1-\log\log 2+\int_2^\infty\frac{\varepsilon(t)}{t\log^2 t}dt-\\ &\qquad -\int_x^\infty\frac{\varepsilon(t)}{t\log^2 t}dt+O\left(\frac{1}{\log x}\right).\end{aligned}$$

Here we have applied the standard method of completing the convergent integral up to infinity. We now bound the tail end by

$$\int_x^\infty \frac{\varepsilon(t)}{t\log^2 t}dt \ll \int_x^\infty \frac{dt}{t\log^2 t}.$$

Change variables to $w = \log t$, so $dw = dt/t$, and the integral becomes

$$\int_{\log x}^\infty \frac{dw}{w^2} = \frac{1}{\log x}.$$

Putting this together we obtain the required result with

$$D = 1 - \log\log 2 + \int_2^\infty \frac{\varepsilon(t)}{t\log^2 t}dt.$$

■

Theorem 2.23 is saying that the following limit exists,

$$D = \lim_{x\to\infty}\left(\sum_{p\le x}\frac{1}{p} - \log\log x\right).$$

The limit is of little practical help, though, in calculating D since the convergence is *extremely* slow. It can be shown that numerically $D \approx 0.261497212....$

The Euler product $\prod_p (1 - 1/p)$ diverges (to 0). In fact, in Problem Sheet 1 you were asked to show that

$$\prod_{p\le x}\left(1 - \frac{1}{p}\right) \le \frac{1}{\log x}$$

as $x \to \infty$. There is a more exact result,

Theorem 2.24 ***Merten***. *There exists a constant $C > 0$ such that*

$$\begin{aligned}\prod_{p\le x}\left(1 - \frac{1}{p}\right) &= \frac{C}{\log x} + O\left(\frac{1}{\log^2 x}\right) \\ &= \frac{C}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right)\end{aligned}$$

as $x \to \infty$.

Proof Taking logarithms of the finite product,

$$\log \prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \sum_{p \leq x} \log \left(1 - \frac{1}{p}\right)$$

$$= -\sum_{p \leq x} \frac{1}{p} + \sum_{p \leq x} \left(\log \left(1 - \frac{1}{p}\right) + \frac{1}{p}\right).$$

As noted in Chapter 1, this latter sum is dominated by

$$\sum_{p \leq x} \frac{1}{p(p-1)},$$

which converges. So we can complete the sum up to infinity (call it b) and bound the error in doing so. That is,

$$\sum_{p \leq x} \left(\log \left(1 - \frac{1}{p}\right) + \frac{1}{p}\right) = b + \sum_{p > x} \left(\log \left(1 - \frac{1}{p}\right) + \frac{1}{p}\right),$$

where it can be shown that $b = -0.315718452...$ .In the last sum use the inequality from Chapter 1,

$$0 < -\log(1-x) - x < \frac{x^2}{(1-x)}$$

for $0 < x < 1$ to get

$$\sum_{p > x} \left|\log \left(1 - \frac{1}{p}\right) + \frac{1}{p}\right| \ll \sum_{p > x} \frac{1}{p(p-1)}$$

$$\leq \sum_{n > x} \left(\frac{1}{n-1} - \frac{1}{n}\right) \leq \frac{1}{x}.$$

Therefore

$$\log \prod_{p \leq x} \left(1 - \frac{1}{p}\right) = -\sum_{p \leq x} \frac{1}{p} + b + O\left(\frac{1}{x}\right)$$

$$= -\log\log x - D + b + \eta(x),$$

by Theorem 2.23, and where $\eta(x)$ is an error term $< C/\log x$ for some $C > 0$. Exponentiate to get

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \frac{1}{\log x} e^{-D+b} e^{\eta(x)}. \tag{23}$$

What does $\exp(\eta(x))$ represent? A result used in the second year analysis course was that

$$|y| < 1/2 \implies |e^y - 1| < 2|y|.$$

If we demand x is so large that $C/\log x < 1/2$, then $|\eta(x)| < 1/2$ in which case

$$\left|e^{\eta(x)} - 1\right| < 2|\eta(x)| = O\left(\frac{1}{\log x}\right),$$

that is

$$e^{\eta(x)} = 1 + O\left(\frac{1}{\log x}\right).$$

Combine to get the stated result with $C = e^{-D+b}$. ■

Note

$$-D + b = -0.2614972128... - 0.315718452... = -0.577215664... ,$$

and

$$C = e^{-D+b} = 0.5614594... .$$

You might think that $-D+b$ looks suspiciously like $-\gamma$, where γ is Euler's constant, and it can be shown that this is so (but not here). This would mean that Theorem 2.24 should really read

$$\prod_{p \le x}\left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right).$$